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Theory of the one-dimensional forest-fire model

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Turbulent cascade processes are studied in terms of a one-dimensional forest-fire model. A hierarchy of steady-state equations for the forests and the holes between them is constructed and solved within a mean-field closure scheme. The exact hole distribution function is found to be $N_H(s)=4N/[s(s+1)(s+2)]$, where N is the number of forests.

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Many physical phenomena can be described as cascade processes where energy, input at large scales, is dissipated at smaller scales. The interval between input and dissipation lengths defines an "inertial range" through which the cascade transfers energy. Rather than being structureless, coherent fractal structures emerge that store and dissipate energy, and give rise to power-law spatial and temporal correlations.

Not only can turbulence in fluids be described as a cascade process [1], many other phenomena may as well. We think of the turbulent cascade as being a unifying concept describing a wide variety of seemingly unrelated processes in nature. Some illustrative phenomena which have been studied using simple microscopic models include, e.g., earthquakes [2], extinction events in evolution [3], volcanic activity [4], and fluctuations in economic activity [5], in addition to fluid turbulence. The observed spatial and temporal scaling behaviors, as well as fractal energy dissipation [6], are thought to be manifestions of dynamical systems where the steady state is built up from the dynamics to be poised at criticality [7]. Although this picture of self-organized criticality has been supported by numerous numerical studies, little analytic progress has been made for these explicitly *nonconservative* models.

Using a forest-fire model in one dimension, we construct a theory for a turbulent cascade directly from microscopic dynamics. A hierarchy of rate equations is determined for the steady state, and these equations are solved using a mean-field closure scheme to obtain the critical exponents characterizing the distribution of activity (fires) and holes between forests. Surprisingly, some of the analytic results obtained are exact.

The forest-fire model considered here is a cellular automaton driven by white noise and defined on a one-dimensional lattice of size L [8]. Each site may be either empty (hole), occupied (tree), or burning (fire). A forest is defined as a connected segment of trees. Forests are separated by gaps which consist of clusters of connected holes. The system is updated in parallel using the following rules.

(i) Trees grow with a small probability p from hole sites at each time step.

(ii) Trees are spontaneously ignited with a small probability f at each time step.

(iii) Any forest that contains a burning tree burns down (becomes empty) in one time step.

The original forest-fire model, proposed by Bak, Chen, and Tang [9] as a model for the spreading of disease or chemical activity, differed in two respects. First, it did not contain a spontaneous ignition parameter f and thus had to be driven above criticality in order for fire to persist. Drossel and Schwabl [8] showed that by introducing a small spontaneous ignition parameter, the system approaches criticality in the limit $f/p\rightarrow 0$, and exhibits scaling in the inertial range $1\ll r\ll\xi$, where $\xi\sim(f/p)^{-\nu}$ is a correlation length. Second, trees burned sequentially so that only nearest-neighbor trees to a tree on fire would burn at the next time step. For this sequential dynamics, an average burning forest of size $\bar{s}$ (defined later) burns down in time $\bar{t}=\bar{s}$. However, at longer time scales, the two dynamics are equivalent as long as $\bar{t}$ is much less than the time scale for a forest to change by trees growing at its edges, i.e., $\bar{t}\ll 1/p$. Instantaneous burning achieves a complete separation between the time scale for burning,which corresponds to avalanches, and the time scale for trees to grow, which corresponds to the driving force, and thus simplifies the analysis that follows.

The rate equations describing how the number of forests [hole clusters] of size s at time t, $N_T(s,t)$ $[N_H(s,t)]$ and other quantities evolve are given below to lowest order in f and p. In one dimension, the total number of forests at time t, $N(t)\equiv\sum_{s=1}^{\infty}N_T(s,t)$, equals the total number of hole clusters, $N(t)=\sum_{s=1}^{\infty}N_H(s,t)$. Given $N(t)$ clusters, the average increment in the number of clusters $\delta N(t)\equiv\langle N(t+1)\rangle-N(t)$, where $\langle\ \rangle$ indicates an average over all stochastic processes, is given by

$$\delta N(t)=-f\sum_{s=1}^{\infty}sN_T(s,t)+p\sum_{s=1}^{\infty}(s-2)N_H(s,t)\ . \qquad (1)$$

The first term on the right-hand side contains contributions from fires igniting on any one of the sites in a forest of size s. The last term contains contributions from trees growing in the interior of hole clusters. We assume that at sufficiently long times the system achieves a steady state where these quantities become stationary, so that $\lim_{t\rightarrow\infty}N_T(s,t)=N_T(s)$, $\lim_{t\rightarrow\infty}N_T(t)=N$, etc. Henceforth we consider only the stationary state, and Eq. (1) becomes

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$$\frac{2N}{L}=1-\rho-\left[\frac{f}{p}\right]\rho\ , \tag{2}$$

where the average density of trees $\rho=1/L\sum_{s=1}^{\infty}sN_T(s)$.

Analogous to fluid turbulence, an infinite hierarchy of equations describes the steady state of forests and hole clusters. First, conservation of the average number of trees, which corresponds roughly to kinetic energy, leads to

$$f\sum_{s=1}^{\infty}s^2N_T(s)=p\sum_{s=1}^{\infty}sN_H(s)\ , \tag{3}$$

where forests of size s burn at a rate of $fsN_T(s)$. The average size of forests that burn, $\bar{s}=\sum_{s=1}^{\infty}s^2N_T(s)/\sum_{s=1}^{\infty}sN_T(s)$, is then

$$\bar{s}=\left[\frac{f}{p}\right]^{-1}\frac{1-\rho}{\rho}\ , \tag{4}$$

as originally found by Drossel and Schwabl [8]. The average size of fires diverges, and the model is critical, as $f/p\rightarrow 0$ as long as $1-\rho$ approaches zero slower than f/p. In this case, Eq. (2) indicates that the number of forests corresponds to the number of holes. Thus, unlike forests, hole clusters have a finite average size (of 2), and only small hole clusters make important contributions to the steady-state dynamics. In particular, isolated holes are important and denoted by $v=N_H(1)/N$.

The stationary hole cluster distribution satisfies

$$-sN_H(s)+2\sum_{s'=s+1}^{\infty}N_H(s')+\left[\frac{f}{p}\right]\sum_{s''=1}^{s-2}\sum_{s'=1}^{s-s''-1}s''\langle\hat{h}(s')\hat{t}(s'')\hat{h}(s-s'-s'')\rangle_{\rm st}=0\ . \tag{5}$$

The first term represents the destruction of hole clusters of size s by growing trees. The second term corresponds to creating hole clusters by growing a single tree in a larger hole cluster. The quantity $\langle\ \rangle_{\rm st}$ represents an average in the stationary state for the joint probability to have forests of size s'' bounded by hole clusters of size s' and $s-s'-s''$. A similar steady-state equation for this joint distribution function of third order leads to terms involving fifth-order distribution functions. Following this procedure generates an infinite hierarchy of presumably exact equations describing neighboring cluster-cluster correlations in the steady state. Fortunately, in the limit $f/p\rightarrow 0$, the nonlinear term in Eq. (5) can be neglected for the important hole clusters which are small, $s\ll s_{\max}\sim(f/p)^{-\nu'}$. Equation (5) can then be written as $-sN_H(s)+(s+3)N_H(s+1)=0$, and has the solution

$$N_H(s)=\frac{4N}{s(s+1)(s+2)}\ . \tag{6}$$

The cutoff for holes $s_{\max}$ can be estimated by ignoring cluster-cluster correlations, so that

$$\langle\hat{h}(s')\hat{t}(s'')\hat{h}(s-s'-s'')\rangle_{\rm st}\rightarrow N_H(s')N_T(s'')N_H(s-s'-s'')/N^2\ . \tag{7}$$

Using the approximate result for the tree distribution, $N_T(s)$, as explained below, leads to $\nu'=2/3$.

Figure 1 demonstrates that our numerical results from simulations of this model agree well with Eq. (6). The simulation was done as follows: on a chain of length L, with periodic boundary conditions, a number $p_0=p/f\ll L$ of random sites were selected. On the empty sites, trees were grown. Then a random site was selected, and if a tree occupied that site, that tree and all trees connected with it were burned. Then, at the next time step, p_0 sites were selected for possible tree growth, and so on. A large number of steps were discarded before the statistics of trees and holes was measured, to ensure that the system had reached a stationary state.

The stationary distribution of forest sizes satisfies

$$\left[1+\frac{f}{2p}s\right]N_T(s)=N_T(s-1)-\langle\hat{h}(1)\hat{t}(s-1)\rangle_{\rm st}+\tfrac{1}{2}\sum_{s'=1}^{s-2}\langle\hat{t}(s')\hat{h}(1)\hat{t}(s-s'-1)\rangle_{\rm st}\ , \tag{8}$$

for $s>1$. in the mean-field approximation (see note added), this can be written as

$$\left[1+\frac{f}{2p}s\right]N_T(s)=(1-v)N_T(s-1)+\frac{v}{2N}\sum_{s'=1}^{s-2}N_T(s')N_T(s-s'-1)\ , \tag{9}$$

and for isolated trees,

$$N_T(1)=\frac{v}{2}+\frac{f}{p}\frac{\rho L}{N}\ . \tag{10}$$

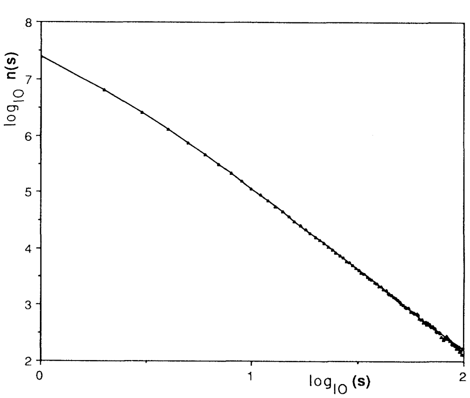

FIG. 1. Distribution of holes calculated numerically (dots) for the system with $L=10^6$, $p_0=10^3$, averaged over 10^5 time steps. The solid line is the analytic result, Eq. (6).

Note that the only contributions from the (noisy) driving force, or tree growth, that enter into these equations are the terms $p(1-\rho)$, which corresponds to uniform growth and growth of trees into isolated holes (pv). Since turbulence in this model is driven by the momentum $q=0$ mode, we suspect that the steady-state dynamics is independent of the specific form of drive, e.g., a different noise spectrum.

For $s<\bar{s}$, neglecting the term with coefficient sf/p in Eq. (9), and introducing a generating function

$$\lambda(z)=\frac{1}{N}\sum_{s=1}^{\infty} z^s N_T(s) \tag{11}$$

into Eq. (9), leads to

$$\lambda(z)=\frac{1-z(1-v)-\sqrt{[z(1-v)-1]^2-(vz)^2}}{vz} . \tag{12}$$

See, for example, Ref. [10] for more details. For large s, the important singularity is the one closest to the origin,

$$\lambda(z)-1=\sqrt{2v(1-v)(1-z)} , \tag{13}$$

and from Eq. (11) follows

$$N_T(s)=\frac{N}{2\pi i}\oint dz\, z^{-s-1}\lambda(z) . \tag{14}$$

Standard methods of contour integration give

$$N_T(s)\sim\int_0^{\infty} e^{-sy}\sqrt{y}\, dy . \tag{15}$$

Combining this with the definition of $\bar{s}$ leads to

$$N_T(s)\sim s^{-\tau}e^{-s/\bar{s}} , \quad \tau=\tfrac{3}{2} . \tag{16}$$

Both the density of holes and the number of forests vanish proportionally as $N\sim 1-\rho\sim(f/p)^{\alpha}$, with $\alpha=\frac{1}{3}$ in the mean-field approximation. This scaling relation together with Eqs. (2) and (3) give the scaling relation $\nu=1/(3-\tau)$. In the mean field $\nu=\nu'=\frac{2}{3}$.

Note added. After submitting this paper, we received a copy of unpublished work by B. Drossel, S. Clar, and F. Schwabl [11] in which they were able to calculate the forest distribution exactly in the limit $f/p\to 0$. They found that the exponent $\tau=2$. After receiving their paper, we found that this result could also be obtained from our cascade equations, e.g., Eq. (8), after realizing that the holes were equally likely to be distributed at any point in a tree-hole sequence of size s.

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[1] A. N. Kolmogorov, Dokl. Acad. Nauk SSR **30**, 299 (1941); M. Nelkin, Science **255**, 566 (1992).
[2] B. Gutenberg and C. F. Richter, Ann. Geofis. **9**, 1 (1956); P. Bak and C. Tang, J. Geophys. Res. B **94**, 15 635 (1989); K. Christensen and Z. Olami, Phys. Rev. A **46**, 1829 (1992).
[3] S. J. Gould, Paleobiology **3**, 135 (1977); M. D. Raup, Science **251**, 1530 (1986); P. Bak and K. Sneppen (unpublished).
[4] P. Diodati, F. Marchesoni, and S. Piazza, Phys. Rev. Lett. **67**, 2239 (1991).
[5] B. Mandelbrot, J. Business **36**, 394 (1963); P. Bak, K. Chen, J. Scheinkman, and M. Woodford, Ric. Economiche **47**, 3 (1993).
[6] B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).
[7] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).
[8] B. Drossel and F. Schwabl, Phys. Rev. Lett. **69**, 1629 (1992).
[9] P. Bak, K. Chen, and C. Tang, Phys. Lett. A **147**, 297 (1990).
[10] P. Bak and H. Flyvbjerg, Phys. Rev. A **45**, 2192 (1992).
[11] B. Drossel, S. Clar, and F. Schwabl (unpublished).